General Case: Eigenvalues and Eigenvectors

We are now ready to tackle the general case of a linear 2×2 system:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy,$$

where the a, b, c, d are constants. We will be following exactly the strategy that we laid out in the previous note. These are key concepts for the rest of the unit, and you should take the time to absorb them.

We want to learn to write the system efficiently in matrix form. So, throughout the derivation, we will give the *expanded* matrix form of our manipulations on the left, and the *abridged* form on the right. For example, our system is:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \Leftrightarrow \qquad \dot{\mathbf{x}} = A\mathbf{x} \qquad\qquad (1)$$

Following the method in the previous note, we look for solutions to our system having the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 e^{\lambda t} \\ a_2 e^{\lambda t} \end{pmatrix} \qquad \Leftrightarrow \qquad \mathbf{x} = e^{\lambda t}\mathbf{a},$$

where a_1, a_2, and λ are unknown constants. We substitute this into the system (1) to determine these unknown constants. Since $D(ae^{\lambda t}) = \lambda a e^{\lambda t}$, we arrive at

$$\lambda e^{\lambda t} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \Leftrightarrow \qquad \lambda e^{\lambda t}\mathbf{a} = e^{\lambda t}A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

We can cancel the factor $e^{\lambda t}$ from both sides, getting

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \Leftrightarrow \qquad \lambda\mathbf{a} = A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

As it stands, we cannot combine the two sides by subtraction, since the scalar λ cannot be subtracted from the square matrix on the right. As in the previously worked example, the trick is to replace the scalar λ by the diagonal matrix λI. This gives

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \Leftrightarrow \qquad \lambda I\mathbf{a} = A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

If we now proceed as we did in the example, subtracting the left side from the right one and using the distributive law for matrix addition and multiplication, we get a 2×2 homogeneous linear system of equations:

$$\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{a} = \mathbf{0}/$$

Written out without using matrices, the equations are

$$\begin{aligned} (a - \lambda)a_1 + ba_2 &= 0 \\ ca_1 + (d - \lambda)a_2 &= 0. \end{aligned} \tag{2}$$

According to the theorem on square homogeneous systems this system has a non-zero solution for the $\mathbf{a}$'s if and only if the determinant of the coefficients is zero, i.e.,

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad |A - \lambda I| = 0.$$

Evaluating the determinant we get a quadratic equation in λ:

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

Definition. This is called the **characteristic equation** of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and if often denoted $p_A(\lambda)$. Its roots λ_1 and λ_2 are called the **eigenvalues** or **characteristic values** of the matrix A.

Remark. In calculating the characteristic equation notice that

$$ad - bc = \det A \qquad a + d = \operatorname{tr} A.$$

Using this, the characteristic equation for a 2×2 matrix A can be written as

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0.$$

In this form, the characteristic equation of A can be written down by inspection; you don't need the intermediate step of writing down $|A - \lambda I| = 0$.

Remark. Abridged vs. expanded notation
In the manipulations above, the matrix notation on the right is compact

to write, which makes the derivation look simpler. On the other hand, its chief disadvantage for beginners is that it is very compressed. Practice writing the sequence of matrix equations so you get some skill in using the notation. Until you acquire some confidence, keep referring to the written-out form on the left, so you are sure you understand what the abridged form is actually saying.

There are now various cases to consider, according to whether the eigenvalues of the matrix A are:
1. two distinct real numbers,
2. a single repeated real number,
3. a pair of conjugate complex numbers.

We begin with the first case: for the rest of this note, the eigenvalues are two distinct real numbers λ_1 and λ_2.

1.   Real distinct eigenvalues

To complete our work, we have to find the solutions to the system (2) corresponding to the eigenvalues λ_1 and λ_2. Formally, the systems become

$$
\begin{array}{ll}
(a - \lambda_1)a_1 + ba_2 = 0 & (a - \lambda_2)a_1 + ba_2 = 0 \\
ca_1 + (d - \lambda_1)a_2 = 0 & ca_1 + (d - \lambda_2)a_2 = 0
\end{array}
\tag{3}
$$

The solutions to these two systems are column vectors, for which we will typically use $\mathbf{v}$.

Definition.   The respective solutions $\mathbf{a} = \mathbf{v}_1$ and $\mathbf{a} = \mathbf{v}_2$ to the systems (3) are called **eigenvectors** (or *characteristic vectors*) corresponding to the **eigenvalues** λ_1 and λ_2.

Remarks.   1. If the work has been done correctly, in each of the two systems in (3), the two equations will be dependent, i.e., one will be a constant multiple of the other. Why? The two values of λ have been selected so that in each case the coefficient determinant $A - \lambda I$ will be zero, which means the equations will be dependent.

2. The solution $\mathbf{v}$ is determined only up to an arbitrary non-zero constant factor: if $\mathbf{v}$ is an eigenvector for λ, then so it $c\mathbf{v}$, for any real constant c; because of this, the line through $\mathbf{v}$ is sometimes called an **eigenline**. A convenient way of finding the eigenvector $\mathbf{v}$ is to assign the value 1 to one of the a_i, then use the equation to solve for the corresponding value of the other a_i. (First try $a_1 = 1$; if that does not work, then $a_2 = 1$ will.)

Once the eigenvalues and their corresponding eigenvectors have been

found, we have two independent solutions to the system (1). They are

$$\mathbf{x}_1(t) \;=\; e^{\lambda_1 t}\mathbf{v}_1, \qquad \mathbf{x}_2(t) \;=\; 2e^{\lambda_2 t}\mathbf{v}_2, \quad \text{where} \quad \mathbf{x}_i(t) \;=\; \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

Definition.   In science and engineering applications, these are usually called the **normal modes**.

Using superposition, the general solution to the system (1) is

$$\mathbf{x} \;=\; c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \;=\; c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 2 e^{\lambda_2 t}\mathbf{v}_2.$$

Remarks.   1. The normal nodes often have physical interpretations; this means that it is sometimes possible to find them just by inspection of the physical problem.

2. In the compact notation, the definitions and derivations are valid for square systems of any size. Thus, for instance, you know how to solve a $3 \, times 3$ system, if its eigenvalues turn out to be real and distinct. We won't consider any such systems in these notes, though.

We will apply these techniques to a worked example in the next note in this session titled *Worked Example: Distinct Real Roots*.

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